HOMOLOGICAL ALGEBRA SOLUTIONS WEEK 1

ERIC Y. CHEN

1(a). Let $i : \mathbf{Z} \to \mathbf{Q}$ be the natural inclusion. To show that i is epi in the category of rings, consider two distinct morphisms of rings $f, g : \mathbf{Q} \to R$ to some target ring R. Since $f \neq g$, there is some rational number $a/b \in \mathbf{Q}$ with $a, b \in \mathbf{Z}$ such that

$$f(a/b) \neq g(a/b)$$

Suppose towards a contradiction that f and g agree on integers; then multiplying the previous inequality by $f(b) = g(b)$ gives

$$f(b)f(a/b) \neq g(b)g(a/b)$$

But the LHS of this inequality is $f(a)$, while the right hand side is $g(a)$, so we have a contradiction.

Let $i : \mathbf{Q} \to \mathbf{R}$ be the natural inclusion. To show that i is epi in the category of Hausdorff topological spaces, consider two distinct continuous maps $f, g : \mathbf{R} \to X$ to some target space X, and suppose $x \in \mathbf{R}$ is a point for which $f(x) \neq g(x)$. Since X is Hausdorff, we can find disjoint open sets $U, V \subset X$ such that $f(x) \in U$ and $g(x) \in V$. Considering their preimages under $f \circ i$ and $g \circ i$, we see that

$$(f \circ i)^{-1}(U) \cap (g \circ i)^{-1}(V) = f^{-1}(U) \cap g^{-1}(U) \cap \mathbf{Q}$$

is nonempty, since $f^{-1}(U) \cap g^{-1}(U)$ is an open set containing x, and $\mathbf{Q} \subset \mathbf{R}$ is dense (by definition of $\mathbf{R}$, for every real number x and every open set W containing x, there is always a rational number inside W). Thus, for any point $q \in (f \circ i)^{-1}(U) \cap (g \circ i)^{-1}(V)$ we have $f(q) \neq g(q)$, hence $f \circ i \neq g \circ i$.

(b). Let $f : A \to B$ be a morphism in the category of groups. If f is an injective set map, then for any two distinct morphisms $g_1, g_2 : C \to A$ we can consider an element $c \in C$ on which $g_1(c) \neq g_2(c)$, and the injectivity of f would imply

$$f(g_1(c)) \neq f(g_2(c))$$

Thus, f is monic. For the opposite direction, suppose f is monic. To see that it is injective, for an arbitrary nonidentity element $a \in A$, consider the two distinct group homomorphisms

$$g_1 : \mathbf{Z} \to A \text{ defined by } g_1(1) = a \text{ and } g_2 : \mathbf{Z} \to A \text{ the trivial homomorphism}$$

Since $f \circ g_1 \neq f \circ g_2$, we see that there is some integer n for which $f(a)^n \neq \mathrm{id}_B$. In particular, $f(a) \neq \mathrm{id}_B$, so f is injective.

2. To see that $\mathcal{A}^I$ is a category, the only nontrivial thing to check is that morphisms in $\mathcal{A}^I$ form a set. For this, we consider two functors $F, G \in \mathcal{A}^I$; by definition, the morphisms between F and G are natural transformations from F to G, i.e., for every $i \in I$ the data of morphisms

$$\eta_i : F(i) \longrightarrow G(i)$$

in $\mathcal{A}$ satisfying some compatibilities as i varies. But such collections $\{\eta_i\}_{i\in I}$ embeds into the set of functions

$$\mathrm{Fun}(I, \sqcup_{i\in I} \mathrm{Hom}_{\mathcal{A}}(F(i), G(i))$$

so the collection of natural transformations from F to G is itself a set.

To see that the Yoneda embedding $h : I \to \mathbf{Sets}^{I^{\mathrm{op}}}$ defined by

$$i \mapsto h_i := \mathrm{Hom}(\,\cdot\,, i)$$

is fully faithful, we need to show that the function

$$\eta : \mathrm{Hom}_I(i, j) \longrightarrow \mathrm{Hom}_{\mathbf{Sets}^{I^{\mathrm{op}}}}(h_i, h_j)$$

induced by h is injective, for every pair of objects $i, j \in I$. For this, consider $f, g \in \mathrm{Hom}_I(i, j)$ two distinct morphisms; then $\eta(f), \eta(g)$ are natural transformations from h_i to h_j and to show they are not equal it suffices to present an object $k \in I$ such that

$$\eta(f) \neq \eta(g) \text{ as functions } h_i(k) = \mathrm{Hom}_I(k, i) \to h_j(k) = \mathrm{Hom}_I(k, j)$$

For this, we consider $k = i$ and the identity map $\mathrm{id}_i \in \mathrm{Hom}_I(i, i)$. Then

$$\eta(f)(\mathrm{id}_i) = f \neq g = \eta(g)(\mathrm{id}_i)$$

as we wanted.

3. We aim to show the adjunction between $\Delta : \mathcal{A} \to \mathcal{A}^I$ and $\lim_{i\in I} : \mathcal{A}^I \to \mathcal{A}$, i.e., a bijection between the sets

$$\mathrm{Hom}_{\mathcal{A}^I}(\Delta(a), F) = \mathrm{Hom}_{\mathcal{A}}(a, \lim_{i\in I} F_i) \tag{1}$$

natural in a and F. By the universal property of $\lim_{i\in I} F_i$, morphisms from a to $\lim_{i\in I} F_i$ is in natural bijection with the set of morphisms from a to each F_i, compatible with the arrows in I. But this is the same as morphisms from the diagonal functor $\Delta(a)$ to F, hence we have (1).

Dually, we want to see an adjunction

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{colim}_{i\in I} F_i, a) = \mathrm{Hom}_{\mathcal{A}^I}(F, \Delta(a)) \tag{2}$$

and the argument is identical. By the universal property of $\mathrm{colim}_{i\in I} F_i$, morphisms from $\mathrm{colim}_{i\in I} F_i$ to a is in natural bijection with the set of morphisms from each F_i to a, compatible with the arrows in I. But this is the same as morphisms from F to the diagonal functor $\Delta(a)$.

4. We need to produce an adjunction

$$\mathrm{Hom}_{\mathcal{B}}(Lx, y) = \mathrm{Hom}_{\mathcal{A}}(x, Ry) \tag{3}$$

natural in x, y, given the data of natural transformations $\eta : \mathrm{id}_{\mathcal{A}} \to RL$ and $\epsilon : LR \to \mathrm{id}_{\mathcal{B}}$.

Starting from the LHS, we consider an arrow

$$(f : Lx \to y) \in \mathrm{Hom}_{\mathcal{B}}(Lx, y)$$

in $\mathcal{B}$. Applying R to this arrow gives an arrow

$$(R(f) : RLx \to Ry) \in \mathrm{Hom}_{\mathcal{A}}(RLx, Ry)$$

Using η, we have an arrow $\eta_x : x \to RLx$, which we can post-compose with $R(f)$ to obtain an arrow

$$(R(f) \circ \eta_x : x \to Ry) \in \mathrm{Hom}_{\mathcal{A}}(x, Ry)$$

This procedure $f \mapsto R(f) \circ \eta_x$ gives a function from the LHS to the RHS of (3). To see that it is a bijection, we consider the analogous procedure in the reverse direction:

$$\mathrm{Hom}_{\mathcal{A}}(x, Ry) \ni g \longmapsto \epsilon_y \circ L(g) \in \mathrm{Hom}_{\mathcal{B}}(Lx, y)$$

which we claim to be inverse to the first. To see this, we compute

$$\begin{aligned}
\epsilon_y \circ L(R(f) \circ \eta_x) &= \left(Lx \xrightarrow{L\eta_x} LRLx \xrightarrow{LRf} LRy \xrightarrow{\epsilon_y} y \right) \\
&= \left(Lx \xrightarrow{L\eta_x} LRLx \xrightarrow{\epsilon_x} Lx \xrightarrow{f} y \right) \text{ by naturality of } \epsilon \text{ with respect to } f \\
&= Lx \xrightarrow{f} y \text{ by assumption on } \eta
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
R(\epsilon_y \circ L(g)) \circ \eta_x &= \left(x \xrightarrow{\eta_x} RLx \xrightarrow{RLg} RLRy \xrightarrow{R\epsilon_y} Ry \right) \\
&= \left(x \xrightarrow{g} Ry \xrightarrow{R\eta_y} RLRy \xrightarrow{R\epsilon_y} Ry \right) \text{ by naturality of } \eta \text{ with respect to } g \\
&= x \xrightarrow{g} Ry \text{ by assumption on } \epsilon
\end{aligned}$$